Slow relaxation in the Ising model on a small-world network with strong long-range interactions

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We consider the Ising model on a small-world network, where the long-range interaction strength J_2 is in general different from the local interaction strength J_1, and examine its relaxation behaviors as well as phase transitions. As J_2/J_1 is raised from zero, the critical temperature also increases, manifesting contributions of long-range interactions to ordering. However, it becomes saturated eventually at large values of J_2/J_1 and the system is found to display very slow relaxation, revealing that ordering dynamics is inhibited rather than facilitated by strong long-range interactions. To circumvent this problem, we propose a modified updating algorithm in Monte Carlo simulations, assisting the system to reach equilibrium quickly.

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When random links are added to a regular lattice, the latter becomes a small-world network, characterized by short path length and high clustering [1,2]. In such a small-world network, the diameter increases very slowly with system size, $l \sim \ln N$, while a regular one displays $l \sim O(N)$. Also having common neighbors for two connected nodes is highly probable. With these features, all elements on the network can exchange information with each other more efficiently than on a regular lattice. Accordingly, it is expected that dynamical systems on small-world networks may display enhanced performance; examples include ordering in spin models [3,4], synchronization in coupled oscillators [5], and computational performance of neural network [6].

A small-world network, constructed from a one-dimensional lattice, has two kinds of connections: (short-range) local links and (long-range) shortcuts. It is conceivable that the two kinds of couplings in a real system have different origins and thus different strengths; this makes it desirable to examine the general case that interactions via local links and via shortcuts are different in strength. It is obvious that in the absence of long-range interactions (via shortcuts), long-range order does not emerge. As a small amount of weak long-range interactions is introduced, however, the system undergoes a phase transition to the state with long-range order. This indicates the importance of shortcuts in ordering, and it is of interest to elucidate how much they are important, relative to local links. On the contrary, without neighbor (local) interactions, the system cannot percolate below a certain value of connectivity. Therefore, we conclude that high clustering due to local interactions is also important for achieving long-range order.

To probe the roles of long- and short-range interactions in ordering, we consider the Ising model as a prototype system exhibiting an order-disorder transition, and examine the transition behavior on a small-world network, varying the long-range interaction strength J_2 relative to the short-range strength J_1. When $J_2=0$, the system reduces to the one-dimensional Ising model and does not display long-range order. In the case of uniform interaction ($J_1=J_2$), the system is known to undergo a phase transition of the mean-field type [3,7,8]. It is expected that the mean-field transition is prevalent for all finite values of J_2/J_1 as long as shortcuts account for a finite fraction of total links. Shortcuts in general assist spins to order, which is reflected by the increase of the critical temperature; similar trends have been found in analytical studies of slightly different systems in *equilibrium* [9,10].

This work focuses on how the system approaches equilibrium and reveals that strong long-range interactions ($J_2/J_1 \gg 1$) give rise to extremely slow relaxation, making Monte Carlo (MC) dynamics based on the Metropolis algorithm inefficient. To avoid such slow convergence, we devise a modified updating algorithm, which assists the system to reach equilibrium more quickly. In the limiting case that $J_1=0$—namely, all nearest-neighbor interactions are deleted—the remaining links (shortcuts) constitute a random network with connectivity kP, where k denotes the range of the local interaction in the underlying lattice and P is the probability of adding or rewiring shortcuts on each local link.

Here the small-world network is constructed in the following way: We first consider a one-dimensional (1D) lattice of N nodes, each of which is connected to its $2k$ nearest neighbors, with k being the local interaction range. Then each local edge is visited once and a random long-range connection (shortcut) is added with probability P (without removing the local edge). Note the difference from the original Watts and Strogatz (WS) construction [2], where local edges are removed and reconnected to randomly chosen nodes.

The Hamiltonian for the Ising model on a small-world network with two such kinds of interactions is given by

$$H = -J_1 \sum_i \sum_{j=1}^{k} \sigma_i \sigma_{i+j} - J_2 \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{1}$$

where $\sigma_i (= \pm 1)$ is the Ising spin on node i of the network. The first term is precisely the Hamiltonian for the 1D Ising model with k nearest neighbors whereas the second one describes the contributions of spin pairs connected via long-range connections.

We perform extensive MC simulations at various values of the addition probability P and the coupling ratio J_2/J_1.

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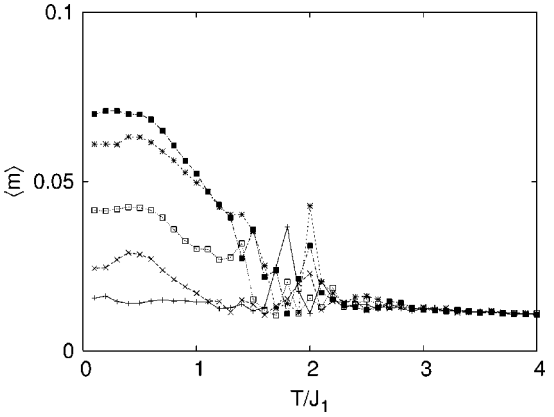

FIG. 1. Erratic behavior of the order parameter with temperature T/J_1 (with the Boltzman constant $k_B \equiv 1$) for $J_2/J_1=10$ and $P=0.1$. The magnetization m has been obtained from the average over 5×10^4 MC steps after the data during initial 5×10^4 MC steps discarded. Data, labeled by five symbols, represent the results of five MC runs, respectively, on a *single* small-world network. The magnetization at low temperatures varies largely run by run, and the system persists to remain in the disordered state.

Specifically, we anneal the system, starting from disordered states at high temperatures, and employ the standard Metropolis algorithm with single-spin-flip updating to compute various quantities including the order parameter (magnetization). As well known, this method is expected to have the system reach efficiently the equilibrium, characterized by the Boltzman distribution, and to give reliable results at all temperatures except in the critical region, where critical slowing down is unavoidable due to strong fluctuations [11]. Obtained are results which in general support the mean-field transition and saturation of the critical temperature, unless long-range interactions are far stronger than local ones.

On the other hand, for J_2 much larger than J_1, the order parameter (magnetization) m turns out to change erratically around the critical temperature and the ordered phase is hardly observed at very low temperatures. Furthermore, at low temperatures it varies largely MC run by run, even though a single small-world network configuration is used (see Fig. 1 for $J_2/J_1=10$ and $P=0.1$; for convenience, the Boltzman constant k_B is set equal to unity throughout this paper). Namely, the result depends upon the random number sequence. This may be explained in the following way: At high temperatures, all spins can flip easily and the system is in the fully disordered state. On a small-world network, there appear clusters which are connected by shortcuts with the interaction strength far larger than the local (nearest-neighbor) one. As the temperature is lowered, spins on such clusters align first along either the up or down direction while other spins on the 1D chain flip easily because thermal fluctuations are still strong compared with local interactions. For the whole spins to be aligned below the critical temperature, all clusters should have the same spin orientation; otherwise, some spins (which do not have long-range interactions) may confuse between spin clusters of different spin directions. However, it is not probable for a spin in the cluster to have opposite directions, due to the strong long-range interactions at such low temperatures. This yields low acceptance ratios in the algorithm, resulting in an extremely long relaxation time. Accordingly, the system tends to remain in a disordered state which does not correspond to the minimum of the free energy, even if the temperature is lower than the critical temperature. Finally, at very low temperatures, spins seldom flip, so that the value of the order parameter depends on the previous history.

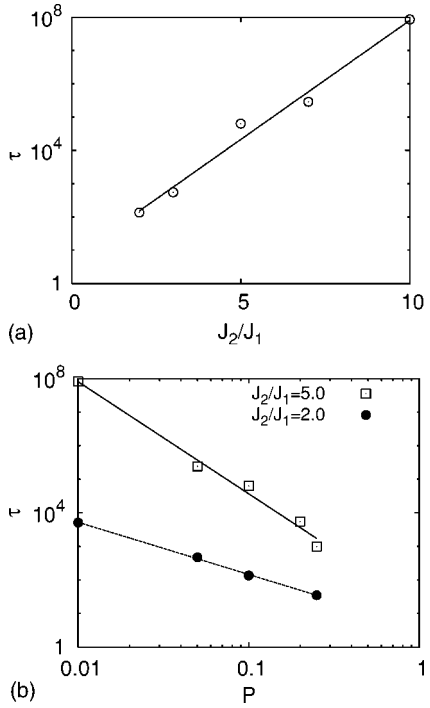

FIG. 2. Relaxation time τ (in units of the MC step) near the critical temperature T_c, estimated from the relation $m-m_{eq}\sim e^{-t/\tau}$, with J_2/J_1 and P varied. (a) Exponential increase of τ with J_2/J_1 for $P=0.1$, reflecting that the updating probability is an exponentially decreasing function of J_2/J_1. The solid line represents the best fit: $\tau=\tau_0 e^{aJ_2/J_1}$ with $\tau_0=1.7$ and $a=1.6$. (b) Algebraic decrease of τ with P for two values of J_2/J_1, which is related to the characteristic path length of the small-world network. The solid and dashed lines correspond to the power-law decay $\tau=\tau_0 P^{-\sigma}$ with $\tau_0=1.4$, $\sigma=1.56$ and $\tau_0=2.8$, $\sigma=3.3$, respectively.

We examine relaxation of the order parameter, starting from the fully ordered state near the critical temperature T_c and from the disordered state at low temperatures, to measure the characteristic time scale for the system to reach equilibrium. Assuming the exponential relaxation in the form $|m-m_{eq}|\sim e^{-t/\tau}$, we estimate the value of τ, varying J_2/J_1 and P. Figure 2 shows the relaxation time τ, measured in units of the MC step, in the system of size $N=6400$ (nominally) at the critical temperature. It is observed that τ grows exponentially from 10^2 to 10^8 as J_2/J_1 is increased. For a given value of J_2/J_1, τ is shown to depend algebraically on P: $\tau\sim P^{-\sigma}$. We stress that these features are not restricted merely to the region near the critical temperature; they persist at all temperatures below the critical temperature, as shown in Fig. 3. In fact they are *even more conspicuous at low temperatures;* this manifests the sharp contrast with the conventional critical slowing down, present only near the critical temperature in systems on regular lattices [11].

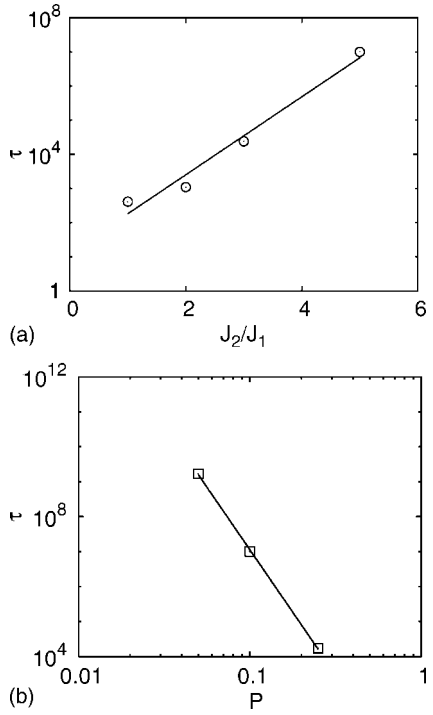

FIG. 3. Relaxation time τ (in units of the MC step) at low temperature $T_c/2$, estimated from the relation $m-m_{eq}\sim e^{-t/\tau}$, with J_2/J_1 and P varied. (a) Exponential increase of τ with J_2/J_1 for $P=0.1$, reflecting that the updating probability is an exponentially decreasing function of J_2/J_1. The solid line represents the best fit: $\tau=\tau_0 e^{aJ_2/J_1}$ with $\tau_0=13$ and $a=2.6$. (b) Algebraic decrease of τ with P for $J_2/J_1=5.0$. The solid line corresponds to the power-law decay $\tau=\tau_0 P^{-\sigma}$ with $\tau_0=0.8$ and $\sigma=7.1$.

One can understand the exponential growth of τ in terms of the inverse updating probability. For large values of J_2/J_1, flipping one spin in a pair which interact strongly with each other will give much influence to the relaxation process. The probability of this update is given by $e^{-\Delta E/T}$ at temperature T, where the energy change $\Delta E=J_2-cJ_1$ depends on the neighboring spin states through integer c. Since the temperature is measured in units of J_1, the inverse of the updating probability leads to the relaxation time in the form $\tau\sim e^{aJ_2/J_1}$, where a is a constant. On the other hand, as the link addition probability P is increased, the characteristic path length l of the system in general reduces in an algebraic way [12]; this allows information to travel more efficiently and thus gives rise to the algebraic decrease of τ with P.

Accordingly, it is concluded that the true equilibrium state may not be obtained within moderate MC steps when long-range interactions are substantially stronger than local ones. To circumvent this problem and to obtain the equilibrium state efficiently, we propose a modified updating method which is efficient in simulations of such a system. The slow relaxation originates from the fact that flipping a spin interacting (strongly) via a shortcut is hardly probable, even though the free energy reduces if accepted. Therefore, when a spin in a cluster linked via shortcuts is selected during sequential update, we also consider, with probability one-half, the possibility of flipping all the spins in the cluster simultaneously. Note that the probability of such cluster updating is much higher than that of usual single-spin updating because the energy difference involves only the short-range interactions. Still single-spin updating is also allowed, so that ergodicity of the system remains intact. Further, the probability to be selected as a cluster is taken always the same for every relevant spin, which guarantees the detailed balance condition. The new algorithm is thus expected to help the system to reach the correct equilibrium quickly, yielding appropriate results efficiently.

To demonstrate the efficiency of the new algorithm, we employ it to probe the case of strong long-range interactions ($J_2/J_1\gtrsim 5$) where the conventional algorithm is practically inapplicable. To find the critical temperature at given values of P and J_2/J_1, we examine the scaling behaviors of the magnetization m, susceptibility χ, specific heat C, and Binder's cumulant [11]. Typically, we consider the system of size up to $N=12\,800$ and take the average over 100 different network realizations as well as the thermal average over 5×10^4 MC steps after equilibration at each temperature.

We write the finite-size scaling forms as $m=N^{-\beta/\bar{\nu}}h(|t|N^{1/\bar{\nu}})$, $\chi=N^{\gamma/\bar{\nu}}g(|t|N^{1/\bar{\nu}})$, and $C=N^{\alpha/\bar{\nu}}f(|t|N^{1/\bar{\nu}})$ with appropriate scaling functions and critical exponents γ, α, β, and $\bar{\nu}$, where $t\equiv(T-T_c)/T_c$ is the reduced temperature. Finite-size scaling analyses of these quantities obtained for $N=1600$, 3200, 6400, and 12 800 unanimously support a phase transition of the mean-field type, with exponents $\gamma=1$, $\alpha=0$, $\beta=1/2$, and $\bar{\nu}=2$. The critical temperature turns out to agree well with the value obtained from the unique crossing point of Binder's cumulant. It is thus concluded that the system undergoes a finite-temperature transition of mean-field nature for $J_2/J_1>0$ and $P\neq 0$.

Here shortcut interactions are essential for the 1D system to display long-range order. The critical temperature T_c/J_1 is expected to increase as J_2/J_1 is raised. In simulations, however, T_c/J_1 does not keep increasing with J_2/J_1 beyond a certain value depending on P. In Fig. 4, we present the phase diagram of the system with range $k=1$, for various values of P. In this case of $k=1$, analytic results have been reported for similar systems: A replica-symmetric solution has been developed on the networks constructed by superimposing random graphs onto a one-dimensional ring [9]. Subsequently, combinatorics has been used to treat quenched disorder on the networks, where each node is restricted from having more than one shortcut [10]. Those networks coincide with our network only in the limit $P\to 0$. For finite P, in contrast to the latter, we allow each node to have more than one shortcut in the construction, which is more realistic and necessary for the small-world network to have an exponential tail in the degree distribution. Further, one end of each added shortcut is determined sequentially, which makes our network have less numbers of large-degree nodes than the former (superimposed random) network. Accordingly, the standard small-world network used in this study lies in between the two types of network in Refs. [9,10]. Since spins on those nodes which have more links facilitate more spins to order, the critical temperature of the system on the small-world network should be lower than that in Ref. [10] and higher that that in Ref. [9], and such a difference is expected to grow as P and J_2 are increased. It is indeed observed in Fig. 4 that the phase boundary of the system on the small-

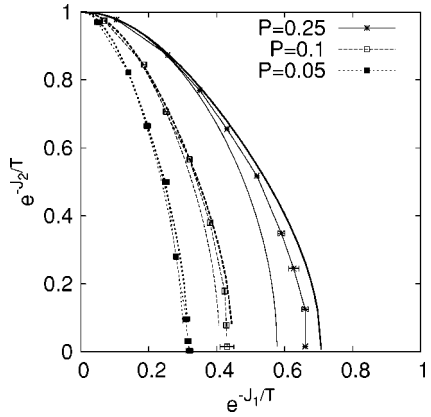

FIG. 4. Phase diagram of the Ising model on a small-world network, where the region below each boundary represents the ordered phase for the corresponding value of the addition probability P. Simulation data for various values of P are depicted by symbols on lines; the latter are merely guides to the eye. Analytic results in Refs. [9,10] are also plotted, with the same kinds of thick and thin lines, respectively, for each value of P. They coincide with numerical results when P is small and/or J_2 is sufficiently smaller than J_1. For J_2/J_1 large, our data locate between the two analytic results in the phase diagram.

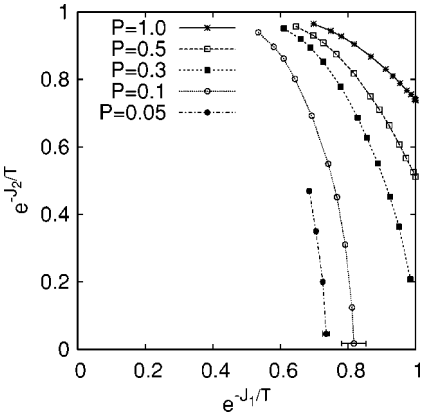

FIG. 5. Phase diagram of the Ising model on a small-world network with range $k=2$. Simulation data for various values of P are depicted by symbols on lines; the latter are merely guides to the eye. For $P \geqslant 0.3$, the phase boundary intersects the $J_1=0$ line at a finite value of J_2, manifesting the presence of a phase transition. This exhibits that a small-world network with local links deleted has a threshold value of P below which no long-range order emerges.

world network locates in between the boundary obtained in Ref. [10] and that in Ref. [9], particularly in case that J_2 is substantially larger than J_1 and P is not very small ($P \gtrsim 0.05$).

We also consider the system with range $k=2$, where local interactions are present between the next-nearest neighbors as well as the nearest neighbors, and perform extensive simulations, the results of which are displayed in Fig. 5. As expected, the region of the ordered phase in the phase diagram is increased compared with the case $k=1$. Except for this, when P is small ($P<0.3$), the overall features are entirely similar to those of the case $k=1$: The critical temperature increases with J_2/J_1, eventually saturating to a finite value. In the case that $P \geqslant 0.3$, on the other hand, one observes an order-disorder transition on the $J_1=0$ line; this corresponds to the small-world network whose local links are all deleted so that there remain only randomly added shortcuts with fraction P. In comparison with the case $P<0.3$, where no ordered phase exists on this line, the percolation problem is manifested in the resulting random graph. Namely, the system is percolating only when its connectivity, given by kP, is higher than $2P_c \approx 0.6$. It is pleasing that this value agrees with the known expression for the threshold value, $P_c=1-\sqrt{(k-1)/k}$ [7]. We have also performed simulations of the system with $k=3$, to obtain fully consistent results. It is of interest that the threshold value is smaller than that of the Erdös-Renyi (ER) random graph [13], which reflects that our random graph is still more regular than the ER graph.

In summary, we have studied via extensive numerical simulations the Ising model on a small-world network, where long-range interactions via shortcuts are in general different from local interactions. It has been demonstrated that long-range interactions via added shortcuts help spins to order, raising the critical temperature at first and having it saturated eventually. Of particular interest is the case of strong long-range interactions, relative to the local ones, where each cluster may play the role of temporarily quenched randomness. The system then tends to be trapped in a local minimum, inhibited from relaxation to the global minimum (i.e., equilibrium); this results in very slow relaxation, making simulations inefficient. This is in contrast with the Ising model on conventional regular or disordered lattices, where severe inhomogeneity in the interaction strength is absent and equilibrium is reached quickly at all temperatures except in the critical region without any erratic behavior. To circumvent this problem, we have developed a modified updating algorithm, assisting the system to reach equilibrium quickly. Any dynamical system on a small-world network with strong long-range interactions is expected to behave similarly, and the modified algorithm developed here may be used to obtain (equilibrium) thermodynamic properties efficiently. Finally, it would be of interest to investigate the case that long- and short-range interactions have opposite signs ($J_2/J_1<0$). The coexistence of ferromagnetic and antiferromagnetic interactions in general introduces frustration into the system, which, together with the randomness associated with the long-range connections, may lead to (truly) glassy behavior [14]. A detailed investigation of how such a glass system relaxes depending on the value J_2/J_1 and comparison with the other cases are left for further study.

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